

GENERALIZED QUATERNIONIC MANIFOLDS

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ABSTRACT

We initiate the study of the generalized quaternionic manifolds by classifying the generalized quaternionic vector spaces, and by giving two classes of nonclassical examples of such manifolds. Thus, we show that any complex symplectic manifold is endowed with a natural (nonclassical) generalized quaternionic structure, and the same applies to the heaven space of any three-dimensional Einstein–Weyl space. In particular, on the product Z of any complex symplectic manifold M and the sphere there exists a natural generalized complex structure, with respect to which Z is the twistor space of M .

INTRODUCTION

There are several natural notions which are similar or generalize the complex manifolds. One of these is based on the idea that, at each point, instead of a single linear complex structure, one considers a quaternionic family of linear complex structures. Then the corresponding integrability condition is in the spirit of Twistor Theory, thus obtaining the basic objects of Quaternionic Geometry (see [7] and the references therein).

On the other hand, one may consider a linear complex structure on the direct sum of the tangent and cotangent bundles, which is orthogonal with respect to the canonical inner product (corresponding to the natural identification of the tangent bundle with its bidual). Then the corresponding integrability condition is provided by a generalization of the usual bracket on vector fields [4], thus leading to the Generalized Complex Geometry (see [5]).

In this note we initiate the natural unification of these two Geometries, under the framework of Twistor Theory.

In Sections 1 and 2 we review the generalized complex and the quaternionic vector spaces, respectively. Then in Section 3 we classify the generalized quaternionic vector spaces (Theorem 3.4). We mention that, although the formulation of this result is elementary, its proof requires a covariant functor which, to any pair formed of a quaternionic vector space and a real subspace, associates a coherent analytic sheaf over the Riemann sphere.

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Finally, in Section 4 we introduce the notion of generalized quaternionic manifold and we present two classes of nonclassical examples, provided by the complex symplectic manifolds (Example 4.2) and by the heaven space of any three-dimensional Einstein–Weyl space (Example 4.3).

1. GENERALIZED COMPLEX VECTOR SPACES

A *generalized linear complex structure* [5] on a (finite dimensional real) vector space V is a linear complex structure on $V \times V^*$ which is orthogonal with respect to the inner product corresponding to the canonical pairing of V and V^* .

Before taking a closer look to the generalized linear complex structures, we recall (see [5]) the *linear B-field transformations*. These can be defined as the orthogonal transformations of $V \times V^*$ which restrict to the identity on V^* . More concretely, any linear B -field transformation is given by $(X, \alpha) \mapsto (X, \iota_X b + \alpha)$, for any $(X, \alpha) \in V \times V^*$, where b is a two-form on V .

Indeed, any linear B -field transformation maps V onto an isotropic complement of V^* , and any such subspace of $V \times V^*$ is the graph of a unique two-form on V (seen as a linear map from V to V^*).

In what follows, we shall use several times this canonical correspondence between linear B -field transformations and isotropic complements of V^* in $V \times V^*$.

Let \mathcal{J} be a generalized linear complex structure on V . There are two extreme cases: $\mathcal{J}V^* = V^*$ and $V^* \cap \mathcal{J}V^* = \{0\}$.

In the first case, we obtain, inductively, that there exists a \mathcal{J} -invariant isotropic complement W of V^* in $V \times V^*$. Consequently, $\mathcal{J}V^* = V^*$ if and only if, up to a linear B -field transformation, we have

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},$$

where J is a linear complex structure on V and we have denoted by J^* the opposite of the transpose of J .

Similarly, on taking $W = \mathcal{J}V^*$ we obtain that $V^* \cap \mathcal{J}V^* = \{0\}$ if and only if, up to a linear B -field transformation, we have

$$\mathcal{J} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix},$$

where ω is a linear symplectic structure on V (seen as a linear isomorphism from V onto V^*).

Note that if \mathcal{J} and \mathcal{K} are linear generalized complex structures on V and W , respectively, then the product $\mathcal{J} \times \mathcal{K}$ is defined, in the obvious way, through the isomorphism $(V \times W) \times (V \times W)^* = (V \times V^*) \times (W \times W^*)$.

Proposition 1.1 ([5]). *Any linear generalized complex structure is, up to a linear B -field transformation, the product of the linear generalized complex structures given by a (classical) linear complex structure and a linear symplectic structure.*

Proof. If \mathcal{J} is a generalized linear complex structure on V then

$$(V^* \cap \mathcal{J}V^*)^\perp = (V^*)^\perp + \mathcal{J}(V^*)^\perp = V^* + \mathcal{J}V^* .$$

Thus, if U is a complement of $V^* \cap \mathcal{J}V^*$ in V^* then $U \cap \mathcal{J}U = \{0\}$ and $U + \mathcal{J}U$ is nondegenerate (with respect to the canonical inner product).

Then the orthogonal complement of $U + \mathcal{J}U$ is a nondegenerate complex vector subspace of $(V \times V^*, \mathcal{J})$ for which $V^* \cap \mathcal{J}V^*$ is a maximal isotropic subspace.

It follows that, up to a linear B -field transformation, (V, \mathcal{J}) is the product of two generalized complex vector spaces (V_1, \mathcal{J}_1) and (V_2, \mathcal{J}_2) satisfying $\mathcal{J}_1 V_1^* = V_1^*$ and $V_2^* \cap \mathcal{J}_2 V_2^* = \{0\}$. \square

The following fact will be used later on.

Lemma 1.2. *Let $\rho : E \rightarrow U$ be a surjective linear map and let $V = U \times (\ker \rho)^*$.*

Then any section of ρ induces an isomorphism $V \times V^ = E \times E^*$ which preserves the canonical inner products. Moreover, any two such isomorphisms differ by a linear B -field transformation.*

Proof. This is a quick consequence of the fact that any section of ρ corresponds to a splitting $E = U \times (\ker \rho)$. \square

2. BRIEF REVIEW OF THE QUATERNIONIC VECTOR SPACES

Let \mathbb{H} be the (unital) associative algebra of quaternions. Its automorphism group is $\mathrm{SO}(3)$ acting trivially on \mathbb{R} and canonically on $\mathrm{Im}\mathbb{H}$.

A *linear hypercomplex structure* on a vector space E is a morphism of associative algebras from \mathbb{H} to $\mathrm{End}(E)$. The automorphism group of \mathbb{H} acts on the space of linear hypercomplex structures on E in an obvious way, thus giving the canonical equivalence relation on it.

A *linear quaternionic structure* on a vector space is an equivalence class of linear hypercomplex structures. A vector space endowed with a linear quaternionic (hypercomplex) structure is a *quaternionic (hypercomplex) vector space*.

Let E be a quaternionic vector space and let $\rho : \mathbb{H} \rightarrow \mathrm{End}(E)$ be a representative of its linear quaternionic structure. Then $Z = \rho(S^2)$ is the space of *admissible* linear complex structures on E . Obviously, Z depends only of the linear quaternionic structure of E .

Let E and E' be quaternionic vector spaces and let Z and Z' be the corresponding spaces of admissible linear complex structures, respectively. A linear map $t : E \rightarrow E'$ is *quaternionic*, with respect to some function $T : Z \rightarrow Z'$, if $t \circ J = T(J) \circ t$, for any $J \in Z$. It follows that if $t \neq 0$ then T is unique and an orientation preserving isometry [7].

The basic example of a quaternionic vector space is \mathbb{H}^k , ($k \in \mathbb{N}$), endowed with the linear quaternionic structure given by its (left) \mathbb{H} -module structure. Furthermore, if E is a quaternionic space, $\dim E = 4k$, then there exists a linear quaternionic isomorphism from \mathbb{H}^k onto E .

The classification (see [10]) of the real vector subspaces of a quaternionic vector space is much harder. It involves two important particular subclasses, dual to each other.

Definition 2.1 ([8]). A *linear CR quaternionic structure* on a vector space U is a pair (E, ι) , where E is a quaternionic vector space and $\iota : U \rightarrow E$ is an injective linear map such that $\text{im } \iota + J(\text{im } \iota) = E$, for any admissible linear complex structure J on E .

A vector space endowed with a linear CR quaternionic structure is a *CR quaternionic vector space*.

By duality, we obtain the notion of *co-CR quaternionic vector space*.

It is convenient to work with the category whose objects are pairs formed of a quaternionic vector space E and a real vector subspace U . Then a CR quaternionic vector space (U, E, ι) corresponds to the pair $(\text{im } \iota, E)$ (and, by duality, a co-CR quaternionic vector space (U, E, ρ) corresponds to the pair $(\ker \rho, E)$).

There exists a covariant functor from this category to the category of coherent analytic sheaves over the sphere. To describe it, let $Z (= S^2)$ be the space of admissible linear complex structures on E and denote by $E^{0,1}$ the holomorphic vector subbundle of $Z \times E^{\mathbb{C}}$ whose fibre, over each $J \in Z$, is the eigenspace of J corresponding to $-i$.

Let τ be the restriction to $E^{0,1}$ of the morphism of trivial holomorphic vector bundles determined by the projection $E \rightarrow E/U$. Denote by \mathcal{U}_- and \mathcal{U}_+ the kernel and cokernel of τ , respectively. Then $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ is the *(coherent analytic) sheaf of (U, E)* [10].

For example, (U, E) is a CR quaternionic vector space if and only if $\mathcal{U} = \mathcal{U}_-$ (which is a holomorphic vector bundle whose Birkhoff–Grothendieck decomposition contains only terms of Chern number at most -1). The simplest concrete examples are (\mathbb{H}, \mathbb{H}) and $(\text{Im } \mathbb{H}, \mathbb{H})$ which correspond to $2\mathcal{O}(-1)$ and $\mathcal{O}(-2)$, respectively; their *duals* are $(0, \mathbb{H})$ and (\mathbb{R}, \mathbb{H}) which correspond to $2\mathcal{O}(1)$ and

$\mathcal{O}(2)$, respectively.

See [8], [9] and [10] for further examples and the related Twistor Theory.

Remark 2.2. 1) If E and E' are quaternionic vector spaces endowed with the real vector subspaces U and U' , respectively, such that either the sheaf of (U, E) or the sheaf of (U', E') is torsion free then [10, Corollary 4.2(i)] the product $(U \times U', E \times E')$ is well-defined, up to a linear quaternionic isomorphism (that is, it does not depend of the particular linear hypercomplex structures, representing the linear quaternionic structures of E and E' , used to define $E \times E'$).

2) Let (U, E) be a pair formed of a quaternionic vector space E and a real vector subspace U ; denote by \mathcal{U} the sheaf of (U, E) .

Then \mathcal{U}_- is the holomorphic vector bundle of a canonically defined CR quaternionic vector space $(U_-, E_-) \subseteq (U, E)$.

Also, there exists pairs (V, F) and (U_t, E_t) such that (V, F) corresponds to a co-CR quaternionic vector space (that is, the projection $E \rightarrow E/V$ defines a linear co-CR quaternionic structure on E/V), the sheaf \mathcal{U}_t of (U_t, E_t) is the torsion subsheaf of \mathcal{U} , and $\mathcal{U}_+ = \mathcal{U}_t \oplus \mathcal{V}$, $(U, E) = (U_-, E_-) \times (U_t, E_t) \times (V, F)$, where \mathcal{V} is the holomorphic vector bundle of (V, F) .

Moreover, the filtration $\{0\} \subseteq (U_-, E_-) \subseteq (U_-, E_-) \times (U_t, E_t) \subseteq (U, E)$ is canonical [10, Corollary 4.2(ii)].

We end this section with the following fairly obvious fact which will be used later on.

Proposition 2.3. *Let (U, E) and (V, F) be pairs formed of a quaternionic vector space and a real subspace such that the sheaf of (U, E) is torsion, whilst (V, F) corresponds to a co-CR quaternionic vector space.*

Then any morphism from (U, E) to (V, F) is zero.

3. GENERALIZED QUATERNIONIC VECTOR SPACES

The notions of linear quaternionic structure and generalized linear complex structure suggest the following.

Definition 3.1. A *generalized linear quaternionic structure* on a vector space V is a linear quaternionic structure on $V \times V^*$ whose admissible linear complex structures are orthogonal with respect to the inner product.

A *generalized quaternionic vector space* is a vector space endowed with a generalized linear quaternionic structure.

There are two basic classes of generalized quaternionic vector spaces.

Example 3.2. To any co-CR quaternionic vector space (U, E, ρ) we associate on $V = U \times (\ker \rho)^*$, by using Lemma 1.2, a generalized linear quaternionic structure which is unique, up to linear B -field transformations.

We, thus, obtain *the generalized quaternionic vector space given by the co-CR quaternionic vector space (U, E, ρ) .*

Note that, this construction gives a (classical) quaternionic vector space if and only if ρ is an isomorphism; equivalently, $V = E$ as quaternionic vector spaces.

By duality, we obtain the generalized quaternionic vector space given by a CR quaternionic vector space.

Example 3.3. Let (V, J, ω) be a vector space endowed with a linear complex structure J and a linear symplectic structure ω . Denote by \mathcal{J}_J and \mathcal{J}_ω the generalized linear complex structures on V given by J and ω , respectively.

Then $\mathcal{J}_J \mathcal{J}_\omega = -\mathcal{J}_\omega \mathcal{J}_J$ if and only if $\omega^{(1,1)} = 0$. Thus, if (V, J, ω) is a complex symplectic vector space then $\{a\mathcal{J}_J + b\mathcal{J}_\omega + c\mathcal{J}_{\omega J} \mid (a, b, c) \in S^2\}$ defines a generalized linear quaternionic structure on V .

We, thus, obtain *the generalized quaternionic vector space given by the complex symplectic vector space (V, J, ω) .*

Note that, if $a \neq \pm 1$ then $a\mathcal{J}_J + b\mathcal{J}_\omega + c\mathcal{J}_{\omega J}$ is given, up to a linear B -field transformation, by a linear symplectic structure.

Here is the main result of this section.

Theorem 3.4. *Any generalized linear quaternionic vector space is, up to a linear B -field transformation, a product of the generalized quaternionic vector spaces given by a CR quaternionic vector space and a finite family of complex symplectic vector spaces; moreover, the factors are unique, up to ordering.*

To prove Theorem 3.4 we, firstly, consider the case when the sheaf of the pair $(V^*, V \times V^*)$ is torsion free.

Proposition 3.5. *Let V be a generalized quaternionic vector space such that the sheaf of $(V^*, V \times V^*)$ is torsion free.*

Then, up to a linear B -field transformation, V is given by a unique (up to isomorphisms) CR quaternionic vector space.

Proof. Let \mathcal{V} be the sheaf of $(V^*, V \times V^*)$ and let \mathcal{V}_\pm be its positive/negative parts.

The canonical inner product induces an isomorphism $\mathcal{V} = \mathcal{V}^*$ which is equal to its transpose and preserves the decompositions into the positive and negative parts. As the positive/negative parts of \mathcal{V}^* are $(\mathcal{V}_\mp)^*$, this corresponds to an

isomorphism $\mathcal{V}_- = (\mathcal{V}_+)^*$.

Thus, under the decomposition $(V^*, V \times V^*) = (U_-, E_-) \times (U_+, E_+)$ into a product of a CR quaternionic and a co-CR quaternionic vector space, the canonical inner product corresponds to an isomorphism $E_- = E_+^*$ with respect to which U_- is the annihilator of U_+ .

Therefore if U'_+ is a complement of U_+ in E_+ then its annihilator corresponds to a complement U'_- of U_- in E_- . Moreover, we may choose U'_+ so that (U'_+, E_+) is a CR quaternionic vector space and, consequently, (U'_-, E_-) is a co-CR quaternionic vector space. Then $U'_- \times U'_+$ is an isotropic complement to V^* defining a linear B -field transformation which is as required. \square

Secondly, we consider the case when the sheaf of $(V^*, V \times V^*)$ is torsion.

Proposition 3.6. *Let V be a generalized quaternionic vector space such that the sheaf of $(V^*, V \times V^*)$ is torsion.*

Then, up to a linear B -field transformation, V is a product of generalized quaternionic vector spaces given by complex symplectic vector spaces; moreover, the factors are unique, up to ordering.

Proof. Let Z be the space of admissible linear complex structures on $V \times V^*$ and let \mathcal{V} be the sheaf of $(V^*, V \times V^*)$.

Suppose, firstly, that the support of \mathcal{V} is formed of two (necessarily, antipodal) points $\pm \mathcal{J} \in Z$; equivalently, $V^* \cap \mathcal{J}(V^*) \neq 0$ and for any $\mathcal{I} \in Z \setminus \{\pm \mathcal{J}\}$, we have $V^* \cap \mathcal{I}(V^*) = \{0\}$.

Let $U = V^* \cap \mathcal{J}(V^*)$ and let $\mathcal{K} \in Z$ be such that $\mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J}$. Then $\mathcal{K}(V^*)$ is a \mathcal{J} -invariant isotropic complement of V^* , and $\mathcal{K}U = \mathcal{K}(V^*) \cap \mathcal{J}(\mathcal{K}(V^*))$.

It follows that, up to a suitable linear B -field transformation, we have that $(V, \mathcal{J}) = (V_1, \mathcal{J}_1) \times (V_2, \mathcal{J}_2)$ such that $V_1 \times V_1^* = U \oplus \mathcal{K}U$, and, also, \mathcal{J}_1 and \mathcal{J}_2 are given by a linear complex and symplectic structures, respectively. Consequently, $E = U \oplus \mathcal{K}U$ is a quaternionic vector subspace of $V \times V^*$ which is nondegenerate with respect to the canonical inner product, and its orthogonal complement is $V_2 \times V_2^*$; in particular, the latter is preserved by \mathcal{K} . Therefore \mathcal{V} contains the sheaf of $(V_2^*, V_2 \times V_2^*)$ which, necessarily, is the sheaf of a co-CR quaternionic vector space. Thus, $V_2 = \{0\}$; equivalently, $\mathcal{J}(V^*) = V^*$.

Then, under the linear B -field transformation corresponding to $\mathcal{K}(V^*)$, we have that \mathcal{J} and \mathcal{K} are given by a linear complex structure J and a linear symplectic structure ω , respectively, which, as \mathcal{J} and \mathcal{K} anti-commute, define a linear complex symplectic structure on V .

In general, the support of \mathcal{V} is $\{\pm \mathcal{J}_k\}_{k=1, \dots, l} \subseteq Z$; denote $U_k = V^* \cap \mathcal{J}_k(V^*)$, $k = 1, \dots, l$. If $\mathcal{I} \in Z \setminus \{\pm \mathcal{J}_k\}_{k=1, \dots, l}$ then from [10, Theorem 3.1] it follows that

we have $U_k \cap \mathcal{I}(U_k) = \{0\}$ and $U_k + \mathcal{I}(U_k)$ is a quaternionic vector subspace of $V \times V^*$, for any $k = 1, \dots, l$; moreover, the sum $\sum_{k=1}^l (U_k + \mathcal{I}(U_k))$ is direct.

Now, inductively (and similarly to the case $l = 1$), we obtain an orthogonal decomposition $V \times V^* = \bigoplus_{k=1}^l (U_k + \mathcal{I}(U_k))$, and the proof follows. \square

We can, now, give the:

Proof of Theorem 3.4. Let Z be the space of admissible linear complex structures on $V \times V^*$ and let \mathcal{V} be the sheaf of $(V^*, V \times V^*)$.

There exists a quaternionic vector subspace E_t of $V \times V^*$ such that, if we denote $U_t = E_t \cap V^*$, then the sheaf of (U_t, E_t) is the torsion part of \mathcal{V} ; in particular, $\dim U_t = \frac{1}{2} \dim E_t$. Consequently, $U_t^\perp = E_t^\perp + V^*$ and $\dim(E_t^\perp \cap V^*) = \frac{1}{2} \dim E_t^\perp$.

By using Proposition 2.3, we obtain that E_t is nondegenerate with respect to the canonical inner product. Hence, E_t^\perp is a nondegenerate quaternionic vector subspace of $V \times V^*$ for which $E_t^\perp \cap V^*$ is a maximal isotropic subspace.

We have thus shown that $(V, V \times V^*) = (E_t^\perp \cap V^*, E_t^\perp) \times (U_t, E_t)$. Therefore, up to a linear B -field transformation, V is a product of two generalized quaternionic vector spaces V_1 and V_2 such that the sheaf of $(V_1^*, V_1 \times V_1^*)$ is torsion free, whilst the sheaf of $(V_2^*, V_2 \times V_2^*)$ is torsion.

By Propositions 3.5 and 3.6, the proof is complete. \square

4. GENERALIZED QUATERNIONIC MANIFOLDS

Recall [5] (see, also, [11]) that a *generalized complex structure* on a manifold M is an orthogonal complex structure on $TM \oplus T^*M$ for which the space of sections of its i -eigenbundle is closed under the Courant bracket [4], defined by

$$[(X, \alpha); (Y, \beta)] = ([X, Y]; \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha)),$$

for any sections (X, α) and (Y, β) of $TM \oplus T^*M$.

A *generalized almost quaternionic structure* on a manifold M is a linear quaternionic structure on $TM \oplus T^*M$ compatible to the canonical inner product.

Let M be a generalized almost quaternionic manifold and let Z be the bundle of admissible generalized linear complex structures on $TM \oplus T^*M$. Note that, Z is the sphere bundle of an oriented Riemannian vector bundle of rank three; in particular, its fibres are Riemann spheres.

Any connection D on $TM \oplus T^*M$, compatible with its linear quaternionic structure and the inner product, induces a connection \mathcal{K} on Z ; in particular, $TZ = \mathcal{K} \oplus (\ker d\pi)$, where $\pi : Z \rightarrow M$ is the projection.

At each $J \in Z$, let \mathcal{J}_J be the direct sum of J and the linear complex structure

of $\ker d\pi_J$, where we have identified $\mathcal{K}_J = T_{\pi(J)}M$, through $d\pi$. Then \mathcal{J} is a generalized almost complex structure on Z .

Definition 4.1. If \mathcal{J} is a generalized complex structure then (M, D) is a *generalized quaternionic manifold* and (Z, \mathcal{J}) is its *twistor space*.

Obviously, the (classical) quaternionic manifolds (see [7] and the references therein) are generalized quaternionic. Also, from [3] it follows that the generalized hyper-Kähler manifolds are generalized quaternionic.

We add the following two classes of examples.

Example 4.2. Let (M, J, ω) be a complex symplectic manifold. Then, by using Example 3.3, we obtain a generalized almost quaternionic structure on M whose bundle of admissible linear complex structure is $M \times S^2$.

Furthermore, let D be any compatible connection on $TM \oplus T^*M$ which induces the trivial connection on $M \times S^2$. Then a straightforward calculation shows that the induced generalized almost complex structure on $M \times S^2$ is integrable. Thus, (M, D) is a generalized quaternionic manifold.

Example 4.3. Let (N^3, c, ∇) be a three-dimensional Einstein–Weyl space and let (M^4, g) be its heaven-space (see [1] and the references therein). Then (M^4, g) is an anti-self-dual Einstein manifold, with nonzero scalar curvature, and ∇ corresponds to a twistorial submersion $\varphi : (M^4, g) \rightarrow (N^3, c, \nabla)$ [6] (see [9]).

As φ is horizontally conformal, at each $x \in M^4$, the differential of φ defines a linear co-CR quaternionic structure on $T_{\varphi(x)}N$ and, hence, φ induces a generalized almost quaternionic structure on M^4 . Recall (Example 3.2) that this is induced by the isomorphism $TM = \mathcal{H} \oplus \mathcal{V}^*$, where $\mathcal{V} = \ker d\varphi$, $\mathcal{H} = \mathcal{V}^\perp$, and we have used the musical isomorphisms determined by g . Furthermore, through this isomorphism, the Levi–Civita connection of (M^4, g) corresponds to a connection D on $TM \oplus T^*M$ with respect to which (M^4, D) is a generalized quaternionic manifold.

The twistor space of (M^4, D) is obtained from the twistor space (Z, J) of (M^4, g) , as follows.

There exists a (pseudo-)Kähler metric \tilde{g} on Z with respect to which the projection $\pi : (Z, \tilde{g}) \rightarrow (M, g)$ is a Riemannian submersion with geodesic fibres (see [2] and the references therein).

On the other hand, φ corresponds to a one-dimensional holomorphic foliation $\tilde{\mathcal{V}}$ on (Z, J) , orthogonal to the fibres of π .

Then $\mathcal{L}(T^{0,1}Z + \tilde{\mathcal{V}}, i\omega)$, where ω is the Kähler form of (Z, J, \tilde{g}) , is the $-i$

eigenbundle of the generalized complex structure \mathcal{J} on Z such that (Z, \mathcal{J}) is the twistor space of (M^4, D) .

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